

# Generalized description of intermittency in turbulence via stochastic methods

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We present a generalized picture of intermittency in turbulence that is based on the theory of stochastic processes. To this end, we rely on the experimentally and numerically verified finding by R. Friedrich and J. Peinke [Phys. Rev. Lett. 78, 863 (1997)] that allows for an interpretation of the turbulent energy cascade as a Markov process of the velocity increments in scale. It is explicitly shown that all known phenomenological models of turbulence can be reproduced by the Kramers-Moyal expansion of the velocity increment probability density function that is associated to a Markov process. We compare the different sets of Kramers-Moyal coefficients of each phenomenology and deduce that an accurate description of intermittency should take into account an infinite number of coefficients. This is demonstrated in more detail for the case of Burgers turbulence that exhibits the strongest intermittency effects. Moreover, the influence of nonlocality on the Kramers-Moyal coefficients is investigated by direct numerical simulations of a generalized Burgers equation. Depending on the balance between nonlinearity and nonlocality, we encounter different intermittency behaviour that ranges from self-similarity (purely nonlocal case) to intermittent behaviour (intermediate case that agrees with Yakhot's mean field theory [Phys. Rev. E 63 026307 (2001)]) to shock-like behaviour (purely nonlinear Burgers case).

**Key words:** Turbulence, energy cascade, anomalous scaling, Markov processes

## 1. Introduction

The phenomenon of homogeneous and isotropic turbulence can still be considered as one of the main unsolved problems in classical physics (Nelkin 1992; Monin & Yaglom 2007). An adequate treatment of the underlying Navier-Stokes equation should make an assertion about the small-scale fluctuations of the longitudinal velocity increments

$$v(\mathbf{x}, \mathbf{r}) = (\mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})) \cdot \frac{\mathbf{r}}{r}, \quad (1.1)$$

in a statistical sense. Here, deviations from Kolmogorov's mean field theory (Kolmogorov 1941) that predicts  $\langle v(\mathbf{x}, \mathbf{r})^n \rangle \sim \langle \varepsilon \rangle^{n/3} r^{n/3}$  are commonly attributed to the intermittent fluctuations of the local energy dissipation rate  $\varepsilon$  and manifest themselves by a non-self-similar probability density function (PDF) of the velocity increments. In turn, this implies a nonlinear order dependence for the scaling exponents  $\zeta_n$  of the moments  $\langle v(\mathbf{x}, \mathbf{r})^n \rangle \sim r^{\zeta_n}$ . In this context, considerable effort has been devoted to the development of phenomenological models of turbulence that all try to account for the intermittent

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character of the local energy dissipation rate such as the infamous log-normal model (Kolmogorov 1962; Oboukhov 1962) or the popular model by She & Leveque (1994) (we also refer the reader to the monograph by Frisch (1995) for further discussion). Despite their success in fitting experimental observations of structure function scaling, these phenomenological models are not obtained from 'first principles', i.e., they are not derived directly from the Navier-Stokes equation.

In this paper we follow a different phenomenological approach (Friedrich & Peinke 1997) that interprets the concept of the turbulent energy cascade, i.e., the transport of energy from large to small scales, as a Markov process of the velocity increments in scale. The vigour of this phenomenology lies in the fact that it is able to reproduce the entire multi-scale velocity increment statistics from the integral length scale down to a scale where the Markov property is violated (Lück *et al.* 2006). The experimentally and numerically verified Markov property of the velocity increments in the inertial range of scales, however, implies that the increment PDF as well as the transition PDF are governed by the same partial differential equation in scale, the so-called Kramers-Moyal expansion. As it is discussed in section 2 of the present paper, the Kramers-Moyal approach allows for a general description of anomalous scaling. Consequently, it is able to reproduce all known phenomenological models of turbulence by the proper choice of the Kramers-Moyal coefficients that enter the Kramers-Moyal expansion.

The result of this paper is that, in order to obtain an accurate description of intermittency effects, higher order Kramers-Moyal coefficients have to be small but non-vanishing. Therefore, the truncation of the Kramers-Moyal expansion as it is done in the usual Fokker-Planck approach (Friedrich & Peinke 1997; Renner 2002; Friedrich *et al.* 2011) might result in an inaccurate description of the tails of the PDFs. To this end, we investigate the asymptotics of the higher-order Kramers-Moyal coefficients of the corresponding phenomenologies in section 2. Section 3 substantiates the existence of higher order coefficients by direct numerical simulations of a generalized Burgers equation.

## 2. Interpretation of the turbulent energy cascade as a Markov process of the velocity increments in scale

In their seminal work, Friedrich & Peinke (1997) investigated the multi-scale velocity increment statistics of a free jet experiment. They could show that the longitudinal velocity increments (1.1) possess a Markov property *in scale*, namely

$$p(v_3, r_3 | v_2, r_2; v_1, r_1) = p(v_3, r_3 | v_2, r_2) \quad \text{for} \quad 0 \leq r_3 \leq r_2 \leq r_1. \quad (2.1)$$

Further experiments (Lück *et al.* 2006) revealed that the Markov property (2.1) is valid in the inertial range and is only broken at small scale separations  $r_2 - r_3 < \lambda_{ME}$ , where  $\lambda_{ME}$  is termed the Markov-Einstein length and is of the order of the Taylor length. An important consequence of the Markov property is that the  $n$ -increment PDF

$$f_n(v_n, r_n; v_{n-1}, r_{n-1}; \dots; v_1, r_1) = \prod_{i=1}^n \langle \delta(v_i - v(\mathbf{x}, r_i)) \rangle, \quad (2.2)$$

can be factorized into products containing only transition probabilities

$$f_n(v_n, r_n; v_{n-1}, r_{n-1}; \dots; v_1, r_1) = p(v_n, r_n | v_{n-1}, r_{n-1}) \dots p(v_2, r_2 | v_1, r_1) f_1(v_1, r_1), \quad (2.3)$$

for all  $r_{i-1} - r_i > \lambda_{ME}$  and  $r_n \leq r_{n-1} \leq \dots \leq r_2 \leq r_1$ . This means a considerable reduction of the complexity of the problem, since the knowledge of the transition probabilities  $p(v_i, r_i | v_{i-1}, r_{i-1})$  is sufficient for the determination of the  $n$ -increment

PDF ( $f_1(v_1, r_1)$  is presumed to be known at large scales). Moreover, a central notion of a Markov process is that the one-increment PDF and the transition PDF follow the same Kramers-Moyal expansion in scale (Risken 1996)

$$-\frac{\partial}{\partial r_1} f_1(v_1, r_1) = \hat{L}_{KM}(v_1, r_1) f_1(v_1, r_1), \quad (2.4)$$

$$-\frac{\partial}{\partial r_2} p(v_2, r_2 | v_1, r_1) = \hat{L}_{KM}(v_2, r_2) p(v_2, r_2 | v_1, r_1), \quad (2.5)$$

where  $\hat{L}_{KM}$  is the Kramers-Moyal operator

$$\hat{L}_{KM}(v, r) = \sum_{k=1}^{\infty} (-1)^k \frac{\partial^k}{\partial v^k} D^{(k)}(v, r), \quad (2.6)$$

and  $D^{(k)}(v, r)$  are the Kramers-Moyal coefficients. Here, the minus signs in Eqs. (2.4 - 2.5) indicate that the process runs from large to small scales. In the following we want to make contact to the scaling solutions of the different phenomenologies of turbulence. To this end, we take the moments of the one-increment PDF in Eq. (2.4)

$$\begin{aligned} -\frac{\partial}{\partial r} \langle v^n \rangle &= -\frac{\partial}{\partial r} \int_{-\infty}^{\infty} dv v^n f_1(v, r) = \sum_{k=1}^{\infty} (-1)^k \int_{-\infty}^{\infty} dv v^n \frac{\partial^k}{\partial v^k} D^{(k)}(v, r) f_1(v, r) \\ &= \sum_{k=1}^n \frac{n!}{(n-k)!} \int_{-\infty}^{\infty} dv v^{n-k} D^{(k)}(v, r) f_1(v, r), \end{aligned}$$

where we dropped the indices of  $v_1$  and  $r_1$ . In order to match powers of  $v$ , we choose  $D^{(k)}(v, r) = \tilde{D}^{(k)}(r) v^k$  and obtain

$$\frac{\partial}{\partial r} \ln \langle v^n \rangle = - \sum_{k=1}^n \frac{n!}{(n-k)!} \tilde{D}^{(k)}(r), \quad (2.7)$$

which is a recurrence relation that can be used to obtain the Kramers-Moyal coefficients of the different models. More precisely, if we assume a scaling of the form  $\langle v^n \rangle \sim r^{\zeta_n}$ , the Kramers-Moyal coefficients read

$$D^{(n)}(v, r) = K_n \frac{(-1)^n v^n}{n! r} \quad \text{and} \quad K_n = \sum_{k=1}^n (-1)^{1-k} \binom{n}{k} \zeta_k. \quad (2.8)$$

The fact that the reduced Kramers-Moyal coefficients  $K_n$  are determined by the scaling exponents  $\zeta_n$  shows that the Kramers-Moyal description is general enough to capture the essence of anomalous scaling. In the next subsections, we will describe in detail how the different phenomenological models can be mapped onto the Kramers-Moyal coefficients.

#### *i.) Kolmogorov's theory K41:*

The monofractal K41 phenomenology (Kolmogorov 1941) states that  $\langle v^n \rangle = C_n \langle \varepsilon \rangle^{n/3} r^{n/3}$  and an evaluation of the reduced Kramers-Moyal coefficients (2.8) suggests that it can be reproduced by just a single Kramers-Moyal coefficient

$$K_n = \begin{cases} 1/3 & \text{for } n \leq 1, \\ 0 & \text{for } n > 1. \end{cases} \quad (2.9)$$

#### *ii.) Kolmogorov-Oboukhov theory K62:*

A first intermittency model which assumes a log-normal distribution of the local rate of energy dissipation  $\varepsilon$  has been proposed by Kolmogorov (1962) and Oboukhov (1962). It

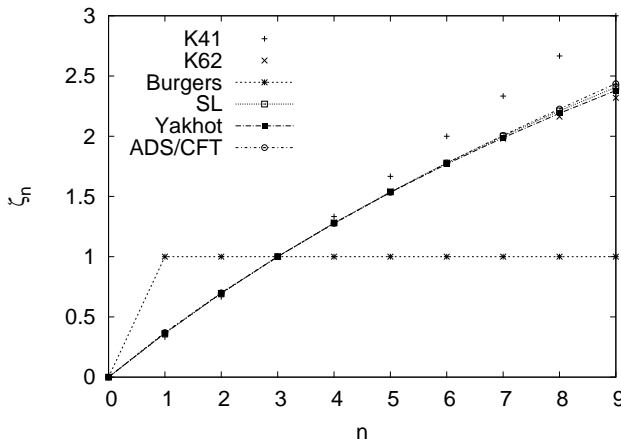


FIGURE 1. Scaling exponents  $\zeta_n$  of velocity structure functions for the different phenomenologies discussed in *i.)-vi.)*. The crosses that are arranged on the straight  $n/3$ -line correspond to the self-similar K41 phenomenology *i.)*. Burgers phenomenology *iii.)* exhibits the strongest intermittency behaviour whereas the other phenomenologies can only be distinguished for higher orders  $n$ . Note that the K62 phenomenology *ii.)* has a parabolic form that violates the structure function convexity condition (Frisch 1995) for  $n \geq \frac{3}{2} + \frac{3}{\mu}$  (not observable in the figure).

predicts the scaling of the structure functions according to  $\langle v^n \rangle = C_n \langle \varepsilon \rangle^{\frac{n}{3}} r^{\frac{n}{3}} \left( \frac{r}{L} \right)^{-\frac{n(n-3)\mu}{18}}$  where  $L$  is the integral length scale and  $\mu$  is the so-called intermittency coefficient which is of the order  $\mu \approx 0.227$ . As it has been discussed by Friedrich & Peinke (1997), this reduces the Kramers-Moyal expansion to a Fokker-Planck equation with drift and diffusion coefficient

$$K_1 = \frac{3+\mu}{9} \quad \text{and} \quad K_2 = \frac{\mu}{18}, \quad (2.10)$$

and implies the vanishing of all higher-order coefficients.

#### *iii.) Burgers scaling:*

The velocity structure functions in Burgers turbulence (Bec & Khanin 2007) follow the extreme scaling

$$\langle v^n \rangle = \begin{cases} C_n \frac{\langle \varepsilon^{n/2} \rangle}{\nu^{n/2}} r^n & \text{for } n < 1, \\ C_n \langle \varepsilon \rangle^{\frac{n}{3}} L^{\frac{n}{3}-1} r & \text{for } n \geq 1. \end{cases} \quad (2.11)$$

Here, the first scaling is due to smooth positive velocity increments in the ramps, whereas the latter scaling corresponds to negative velocity increments dominated by shocks that form due to the compressibility of the velocity field in the vicinity of the viscosity  $\nu \rightarrow 0$ . The smooth solutions correspond to a single Kramers-Moyal coefficient, whereas the shock solutions can only be reproduced by an infinite number of Kramers-Moyal coefficients and we obtain

$$\begin{aligned} K_1 &= 1, \quad K_n = 0 \quad \text{for } n > 1, & \text{for positive increments.} \\ K_n &= 1, \quad \forall n & \text{for negative increments.} \end{aligned} \quad (2.12)$$

#### *iv.) She-Leveque model:*

The She-Leveque model (She & Leveque 1994) for 3D Navier-Stokes turbulence predicts scaling exponents  $\zeta_n = \frac{n}{9} + 2 \left( 1 - \left( \frac{2}{3} \right)^{n/3} \right)$  that are in very good agreement with both experimental and numerical data. This yields an infinite set of coefficients (Nickelsen

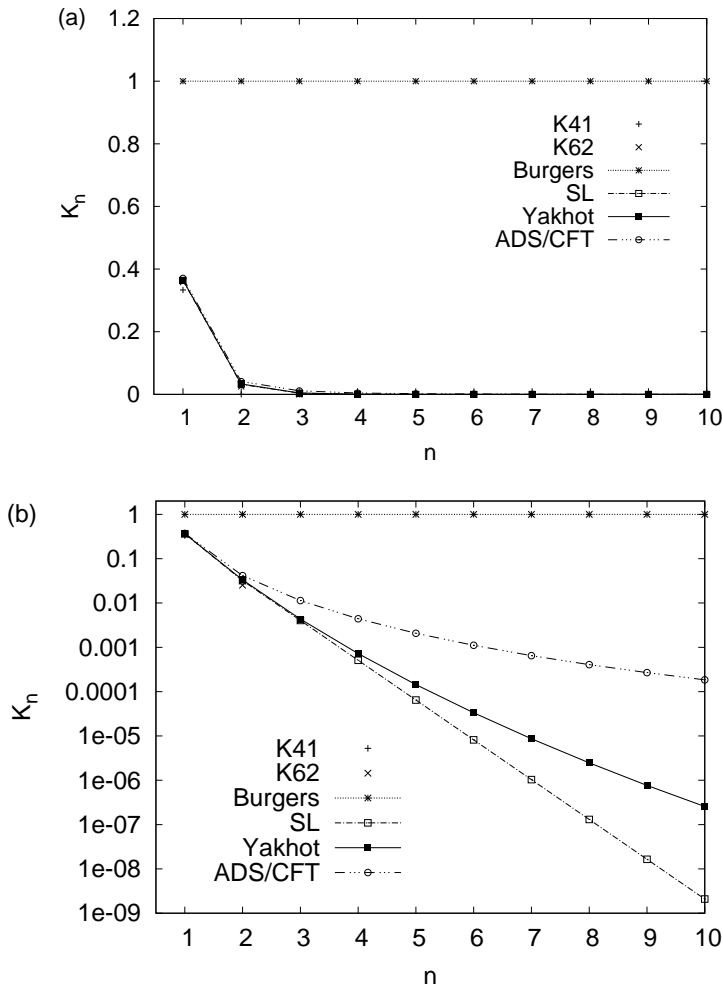


FIGURE 2. (a) Reduced Kramers-Moyal coefficients from Eq. (2.8) for different phenomenological models of turbulence up to the order  $n = 10$ . Coefficients for  $n > 2$  seem to tend towards zero. (b) Semi-logarithmic plot of the reduced Kramers-Moyal coefficients. All phenomenological models except for K41 and K62 show an asymptotic behaviour. Note that the She-Leveque model possesses a nearly linear slope in the semi-logarithmic representation.

2015) and the reduced Kramers-Moyal coefficients read

$$K_n = \frac{n}{9} {}_1F_0(1 - n; ; 1) + 2 \left( 1 - \sqrt[3]{\frac{2}{3}} \right)^n, \quad (2.13)$$

where  ${}_nF_n(a; b; z)$  is the generalized hypergeometric function.

*v.) Yakhot model:*

Yakhot (2001, 2006) introduced a model for structure function exponents  $\zeta_{2n} = \frac{2(1+3\beta)n}{3(1+2\beta n)}$  based on a mean-field approximation. With the choice of  $\beta = 0.05$ , structure functions agree equally well with experimental data as the popular She-Leveque model. The

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run	#1 ( $\alpha = 1$ )	#2 ( $\alpha = 0$ )	#3 ( $\alpha = 0.15$ )
$u_{rms}$	0.0079	0.0058	0.0026
$\nu$	0.000014	0.00001	$1 \times 10^{-6}$
$\langle \varepsilon \rangle$	$5.45 \times 10^{-7}$	$1.38 \times 10^{-7}$	$6.23 \times 10^{-8}$
dx	0.002	0.0015	0.0015
$\eta$	0.0084	0.0092	0.002
$\lambda$	0.0401	0.04895	0.0104
$Re_\lambda$	22.71	28.1668	27.36
$L$	0.9119	1.379	0.286
$T$ in $T_L$	7441	1057	2299
$N$	3072	4096	4096
cut-off	3052	4066	4066

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TABLE 1. Characteristic parameters of the numerical simulations: root mean square velocity  $u_{rms} = \sqrt{\langle u^2 \rangle}$ , viscosity  $\nu$ , averaged rate of local energy dissipation  $\langle \varepsilon \rangle = 2\nu \left\langle \left( \frac{\partial u}{\partial x} \right)^2 \right\rangle$ , grid spacing dx, dissipation length  $\eta = \left( \frac{\nu^3}{\langle \varepsilon \rangle} \right)^{1/4}$ , Taylor length  $\lambda = u_{rms} \sqrt{\frac{\nu}{\langle \varepsilon \rangle}}$ , Taylor-Reynolds number  $Re_\lambda = \frac{u_{rms} \lambda}{\nu}$ , integral length scale  $L = \frac{u_{rms}^3}{\langle \varepsilon \rangle}$ , large-eddy turn-over time  $T_L = \frac{L}{u_{rms}}$ , number of grid points  $N$  and cut-off of the power law forcing. The intermediate case ( $\alpha = 0.15$ ) included a damping term of the form  $-\gamma u(x, t)$  with  $\gamma = 0.03$  on the r.h.s. of Eq. (3.1)

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translation to the Kramers-Moyal coefficients is given by

$$K_n = \frac{\Gamma[n+1]}{\Gamma\left[n+1+\frac{1}{\beta}\right]} \left( \Gamma\left[1+\frac{1}{\beta}\right] + \frac{1}{3\beta^2} \Gamma\left[\frac{1}{\beta}\right] \right). \quad (2.14)$$

*vi.) ADS/CFT random geometry model:*

Eling & Oz (2015) introduced a structure function scaling model which is motivated by a gravitational Knizhnik-Polyakov-Zamolodchikov (KPZ)-type relation. For 3D Navier-Stokes turbulence, they derive

$$\zeta_n = \frac{\left( (1+\gamma^2)^2 + 4\gamma^2 \left( \frac{n}{3} - 1 \right) \right)^{\frac{1}{2}} + \gamma^2 - 1}{2\gamma^2}, \quad (2.15)$$

where experimental data suggests the value  $\gamma^2 = 0.161$ . Unfortunately, we could not obtain an analytical formula for the coefficients of this particular model and have restricted ourselves to a numerical evaluation of Eq. (2.8).

We have plotted the reduced Kramers-Moyal coefficients  $K_n$  for the different models up to the order  $n = 10$  in Fig. 2 (a). As one can see, all models besides K41 and Burgers can hardly be distinguished from one another and the reduced Kramers-Moyal coefficients seem to tend towards zero very quickly. According to a theorem due to Pawula (1967) (see also (Risken 1996)), the vanishing of the fourth-order Kramers-Moyal coefficient implies that all higher coefficients are zero as well and the Kramers-Moyal expansion (2.4) reduces to an ordinary Fokker-Planck equation. The latter is particularly suitable for modelling approaches via its corresponding Langevin equation as well as the undemanding determination of statistical quantities via the exact short-scale propagator of the Fokker-Planck equation (Risken 1996). In the original work (Friedrich & Peinke 1997) and also all subsequent works (Lück *et al.* 2006; Renner *et al.* 2001; Renner 2002) it was argued in favour of Pawula's theorem since the experimentally determined Kramers-Moyal coefficient of order four was very close to zero. Fig. 2 (a) seems

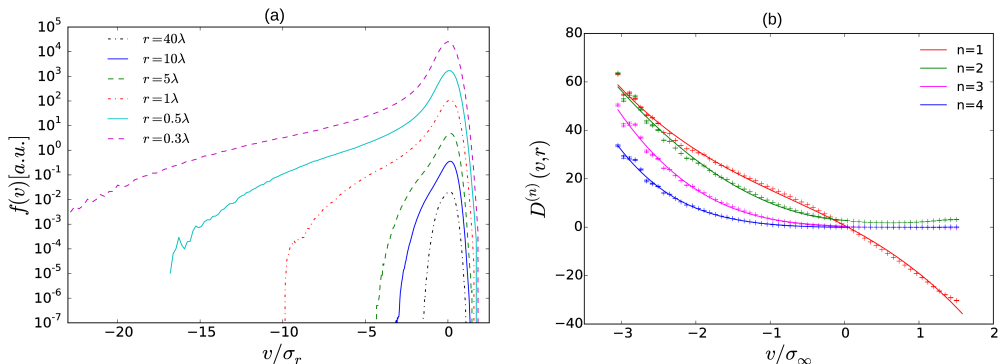


FIGURE 3. (a) Evolution of the velocity increment PDF in scale for the Burgers case  $\alpha = 1$ . The PDFs are shifted vertically and normed with their corresponding standard deviation  $\sigma_r$  for improved visualization. The pronounced left part of the PDFs is dominated by small-scale shock events whereas the right part exhibits nearly self-similar behaviour. (b) Estimation of the Kramers-Moyal coefficients from DNS of Burgers turbulence for  $r = L/2$ . The fits correspond to polynomials of the order  $n$  of the coefficient except for  $n = 1$  where a polynomial of order three has been used. The reduced Kramers-Moyal coefficients have been determined according to  $K_1 = 1.1689 \pm 0.0811$ ,  $K_2 = 0.7880 \pm 0.1406$ ,  $K_3 = 0.6956 \pm 0.1731$  and  $K_4 = 0.7137 \pm 0.1200$

to agree qualitatively with this finding. However, in order to demonstrate that this can be misleading, we show a semi-logarithmic plot of Fig. 2 (a) in Fig. 2 (b). It can be seen that the models *iv.)-vi.)* tend asymptotically towards zero and higher-order Kramers-Moyal are rather small but strictly non-zero. At this point, we want to emphasize that since  $K_4 \approx 10^{-3}$ , the significant detection of these higher-order coefficients in the experiment might be quite challenging due to the presence of measurement noise or insufficient statistics. Nevertheless, since the models *iv.)-vi.)* agree quite well with experimental data an accurate determination of the higher-order coefficients should be within the reach of a spatially and temporarily well-resolved high-Reynolds number experiment. Moreover, Pawula's theorem directly reduces the velocity increment statistics to families of the K62 phenomenology *ii.)*. It should therefore be noted that the latter is only valid for moments  $\langle v^n \rangle$  that do not exceed the order  $n \geq \frac{3}{2} + \frac{3}{\mu}$ , due to the convexity condition for  $\zeta_{2n}$  (see also (Frisch 1995) for further discussion). Consequently, one should bear in mind that whilst modelling or other purposes of the Fokker-Planck approach, the tails of the PDFs might not be described accurately, although - admittedly - this effect should be rather small.

### 3. Detection of higher-order Kramers-Moyal coefficients in direct numerical simulations of a generalized Burgers equation

In order to validate this generalized picture of intermittency and to classify the occurrence of higher-order Kramers-Moyal coefficients, we have performed direct numerical simulations (DNS) of Burgers turbulence. The fact that the forced 1D Burgers equation exhibits rather strong intermittency effects and offers a sufficient quantity of data makes

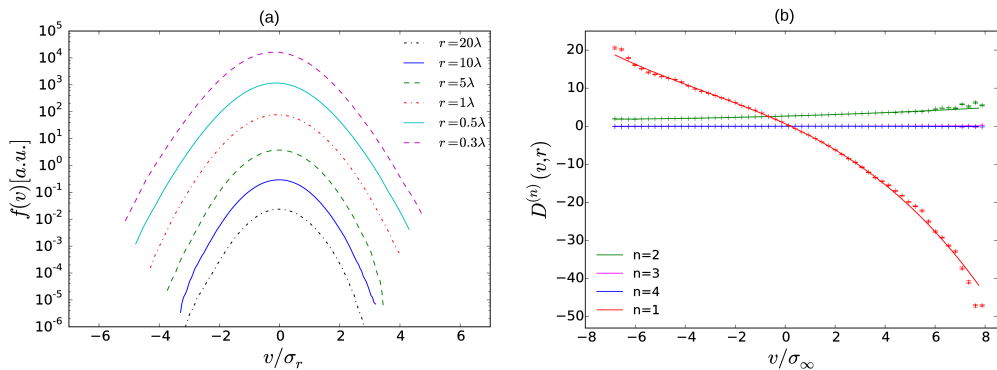


FIGURE 4. (a) Evolution of the velocity increment PDF in scale for the purely nonlocal case  $\alpha = 0$ . The PDFs exhibit self-similarity in the inertial range. (b) Estimation of the Kramers-Moyal coefficients from DNS of the purely nonlocal case Burgers turbulence  $\alpha = 0$ . The fits correspond to polynomials of the order  $n$  of the coefficient except for  $n = 1$  where a polynomial of order three has been used. The reduced Kramers-Moyal coefficients have been determined according to  $K_1 = 0.3108 \pm 0.0002$ ,  $K_2 = 0.0021 \pm 0.0001$ ,  $K_3 = (2.64 \pm 0.01) \times 10^{-5}$  and  $K_4 = (2.2839 \pm 2.56) \times 10^{-5}$ .

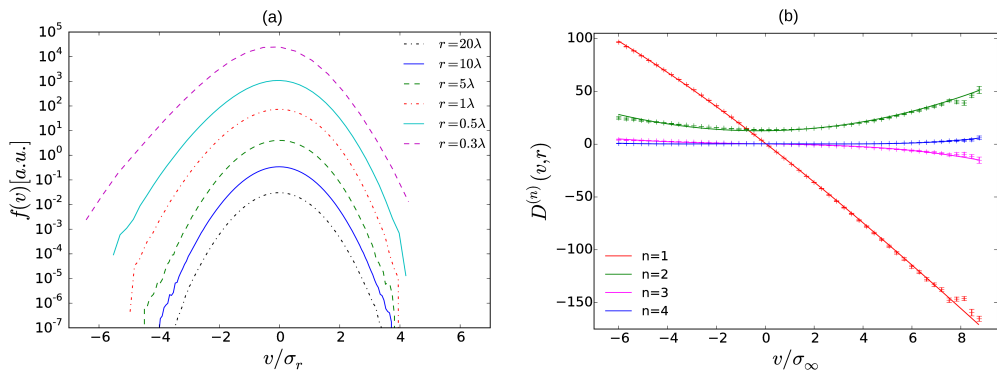


FIGURE 5. (a) Evolution of the velocity increment PDF in scale for the intermediate case  $\alpha = 0.15$ . The PDFs show a slight asymmetry at small scales. (b) Estimation of the Kramers-Moyal coefficients from DNS of the intermediate case  $\alpha = 0.15$ . The fits correspond to polynomials of the order  $n$  of the coefficient except for  $n = 1$  where a polynomial of order three has been used. The reduced Kramers-Moyal coefficients have been determined according to  $K_1 = 0.4356 \pm 0.0007$ ,  $K_2 = 0.0208 \pm 0.0004$ ,  $K_3 = 0.0014 \pm 0.0001$  and  $K_4 = 0.00041 \pm 0.00001$ .



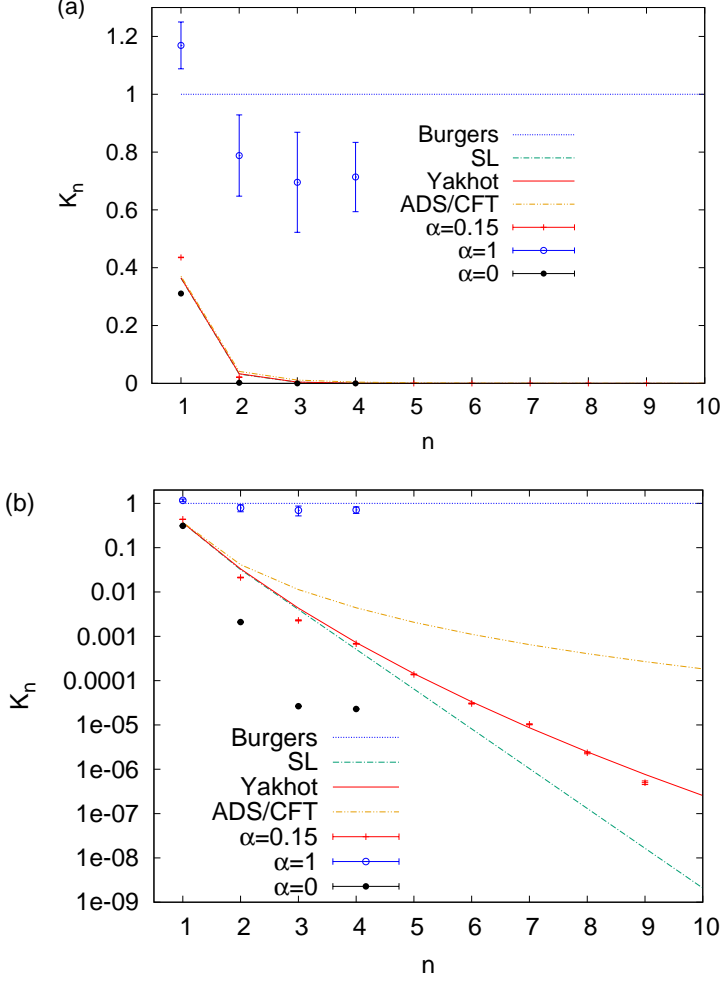


FIGURE 6. (a) Reduced Kramers-Moyal coefficients for DNS of the generalized Burgers equation (3.1). The coefficients for Burgers turbulence ( $\alpha = 1$ ) are in the range of 1 which agrees with the phenomenological predictions. The intermediate ( $\alpha = 0.15$ ) and the purely nonlocal case ( $\alpha = 0$ ) can hardly be distinguished from one another in this representation. The semi-logarithmic plot (b), however, reveals that the reduced Kramers-Moyal coefficients for the intermediate case follow Yakhot’s mean field theory *v.*). Note that higher-order  $n > 5$  coefficients for the Burgers and the purely nonlocal case could not be accurately obtained due to poor polynomial fits (Burgers case) or considerable deviations from Eq. 2.8 (nonlocal case).

it an ideal candidate for a possible detection of higher-order Kramers-Moyal coefficients. Furthermore, the influence of nonlocality can be included by considering the generalized Burgers equation

$$\frac{\partial}{\partial t}u(x, t) + w(x, t)\frac{\partial}{\partial x}u(x, t) = \nu\frac{\partial^2}{\partial x^2}u(x, t) + F(x, t), \quad (3.1)$$

where the convective velocity field is given by

$$w(x, t) = \alpha u(x, t) + (1 - \alpha)\text{p.v.} \int dx' \frac{u(x', t)}{x - x'}. \quad (3.2)$$

Here,  $\alpha = 1$  corresponds to the case of Burgers turbulence, whereas  $\alpha = 0$  corresponds to the purely nonlocal case that is dominated by self-similar behaviour (Zikanov *et al.* 1997). The forcing procedure possesses a power-law in Fourier space  $\langle \hat{F}(k, t) \hat{F}(k', t) \rangle \sim k^{-1} \delta(k - k') \delta(t - t')$  as it has been discussed by Chekhlov & Yakhot (1995) as well as by Zikanov *et al.* (1997). Furthermore, we used a de-aliasing filter in Fourier space (Hou & Li 2007). The relevant turbulent length and time scales of the simulations can be found in Table 1.

We have estimated the Kramers-Moyal coefficients for  $r = L/2$  via an extrapolation method for the conditional moments along the lines of Renner (2002). Here, the  $v$ -range has been rescaled with  $\sigma_\infty = \sqrt{2}u_{rms}$  for better comparison. The obtained coefficients for Burgers turbulence ( $\alpha = 1$ ) in Fig. 3 (b) agree quite well with the theoretical predictions (2.12). Higher-order coefficients ( $n = 3, 4$ ) can be detected significantly for negative increments due to rare large-negative gradient events. Moreover, only the drift coefficient is different from zero for positive increments whereas all higher order coefficients drop to zero for  $v > 0$ . It should be noted that  $D^{(2)}$  possesses an additional intercept that is due to the non-conservative forcing procedure. The reduced Kramers-Moyal coefficients have been obtained via polynomial fits (see caption in Fig. 3). Moreover, the evolution of the one-increment PDF in scale is depicted in Fig. 3 (a) and shows a pronounced left tail due to shock events (Balkovsky *et al.* 1997; E & Vanden Eijnden 1999).

Concerning the purely nonlocal case ( $\alpha = 0$ ), we observe a self-similar evolution of the one-increment PDF in scale which can be seen from Fig. 4 (a). As self-similar behaviour is characterized by a single drift coefficient  $D^{(1)}$ , higher order Kramers-Moyal coefficients should be close to zero. In fact, Fig. 4 (b) shows that  $D^{(3)}$  and  $D^{(4)}$  are rather small. Furthermore, the diffusion coefficient  $D^{(2)}$  is linear in  $v$  and has only a small reduced Kramers-Moyal coefficient  $K_2 = 0.0021 \pm 0.0001$ . Finally,  $K_1 = 0.3108 \pm 0.0002$  suggests that the purely nonlocal case can be described quite accurately by the K41 theory *i.*), which has already been reported by Zikanov *et al.* (1997). Turning to the intermediate case  $\alpha = 0.15$ , Fig. 5 (b) indicates that  $D^{(2)}$  shows a pronounced parabolic form in contrast to the aforementioned purely nonlocal case. In addition, higher order coefficients  $D^{(3)}$  and  $D^{(4)}$  possess also a slight cubic and quartic  $v$ -dependence. Obviously, the latter coefficients are rather small compared to the Burgers case in Fig. 3 (b). In order to discuss this behaviour quantitatively, we have added the numerically obtained reduced Kramers-Moyal coefficients to the phenomenological predictions in Fig. 2 (a) and (b). Fig. 6 (b) reveals that the reduced Kramers-Moyal coefficients up to order 9 possess an asymptotic behaviour that is consistent with Yakhot's mean field theory (Yakhot 2001).

#### 4. Conclusion and Outlook

The present paper underlines the importance of the multi-scale approach devised by Friedrich & Peinke (1997) which is capable of capturing the general effects of anomalous scaling in turbulence embodied in Eq. (2.8). An admissible description of intermittency in turbulence, however, should take into account an infinite number of Kramers-Moyal coefficients, which has been demonstrated by the semi-logarithmic plots of the reduced Kramers-Moyal coefficients in Fig. 2 (b) and 6 (b). Further work will be dedicated to the investigation of higher-order Kramers-Moyal coefficients in the experiment and in DNS of 3D turbulence. In this context, the presented semi-logarithmic plot in Fig. 6 (b) might as well be a more accurate method for the determination of possible scaling behaviour than the usual structure function plot Fig. 1. This would open the possibility to decide which of the various phenomenological models is best suited to describe 3D Navier-Stokes turbulence. In the case of artificial generalized Burgers turbulence, the

Yakhot’s mean field model (Yakhot 2001) could clearly be confirmed as the most accurate candidate. Moreover, the Kramers-Moyal approach should yield important insights in the ongoing discussion about different intermittency behaviour between longitudinal and transverse structure functions. Here, the simple rescaling relation between longitudinal and transverse structure functions (Grauer *et al.* 2012) might be extended to allow for different intermittency in tuning the corresponding set of reduced Kramers-Moyal coefficients in Eq. (2.8).

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